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ON THE REMOVAL OF TERMS FROM AN EQUATION OF THE FIFTH DEGREE.

BY W. E. HEAL, WHEELING, IND.

LET us take the equation $x^5+p_1x^4+p_2x^3+p_3x^2+p_4x+p_5=0$. We propose to remove any *three* of the mean terms and then reduce the equation to any of the forms $x^5+q_4x+q_5=0$, $x^5+q_3x^2+q_5=0$, $x^5+q_2x^3+q_5=0$, or $x^5+q_1x^4+q_5=0$.

Let $y=a+bx+cx^2+dx^3+ex^4$. We wish to eliminate x and thus form an equation in y. Since for every value of x there is a corresponding value for y, the equation in y will be of the fifth degree.

Denote the roots of the equation $x^5+p_1x^4+p_2x^3+p_3x^2+p_4x+p_5=0$ by x_1, x_2, x_3, x_4, x_5 and the corresponding values of y by y_1, y_2, y_3, y_4, y_5.

Then we have

$$\left.\begin{aligned} y_1 &= a+bx_1+cx_1^2+dx_1^3+ex_1^4 \\ y_2 &= a+bx_2+cx_2^2+dx_2^3+ex_2^4 \\ y_3 &= a+bx_3+cx_3^2+dx_3^3+ex_3^4 \\ y_4 &= a+bx_4+cx_4^2+dx_4^3+ex_4^4 \\ y_5 &= a+bx_5+cx_5^2+dx_5^3+ex_5^4 \end{aligned}\right\} \tag{1}$$

Let $\Sigma(x^n)=x_1^n+x_2^n+x_3^n+x_4^n+x_5^n$; $\Sigma(y^n)=y_1^n+y_2^n+y_3^n+y_4^n+y_5^n$.

Adding the five equations of (1) we obtain

$$\Sigma(y)=5a+b\Sigma(x)+c\Sigma(x^2)+d\Sigma(x^3)+e\Sigma(x^4). \tag{2}$$

Also squaring the equations of (1) and adding,

$$\Sigma(y^2)=5a^2+2ab\Sigma(x)+(2ac+b^2)\Sigma(x^2)+2(ad+bc)\Sigma(x^3)+(2ae+2bd+c^2)\Sigma(x^4)+2(be+cd)\Sigma(x^5)+(2ce+d^2)\Sigma(x^6)+2de\Sigma(x^7)+e^2\Sigma(x^8). \tag{3}$$

In a similar manner we can express $\Sigma(y^3)$, $\Sigma(y^4)$ and $\Sigma(y^5)$ by a homogeneous equation in a, b, c, d, e, since $\Sigma(x^n)$ can be expressed in terms of p_1, p_2, p_3, p_4, p_5.

Then, if we denote the coefficients of the equation in y, by P_1, P_2, P_3, P_4, P_5, we have, by Newton's Theorem,

$$\Sigma(y^m)+P_1\Sigma(y^{m-1})+P_2\Sigma(y^{m-2})+\ldots+P_{m-1}\Sigma(y)+mP_m=0,$$

where m may be any integer less than 5.

From this we easily see that if $\Sigma(y)$, $\Sigma(y^2)$ and $\Sigma(y^3)$ be put equal to zero the equation will be reduced to the form $y^5+q_4y+q_5=0$, and if $\Sigma(y)$, $\Sigma(y^2)$ and $\Sigma(y^4)$ all equal zero the equation will be of the form $y^5+q_3y^2+q_5=0$. But if $\Sigma(y)=0$ we obtain from (2) $5a=-[b\Sigma(x)+c\Sigma(x^2)+d\Sigma(x^3)+e\Sigma(x^4)]$. Substituting this in (3) we will have a homogeneous equation in b, c, d and e; denote it by

$$A_1b^2+2A_2bc+2A_3bd+2A_4be+A_5c^2+2A_6cd+2A_7ce+A_8d^2+2A_9de+A_{10}e^2=\text{function }(b, c, d, e). \tag{4}$$

Now assume
function $(b, c, d, e) = lb + mc + nd + pe)^2 + (qc + rd + se)^2 + (td + ne)^2 + (ve)^2$.

Expanding and equating to the corresponding coefficients of (4),

$l^2 = A_1$ (6); $lm = A_2$ (7); $ln = A_3$ (8); $lp = A_4$ (9); $m^2 + q^2 = A_5$ (10); $mn + qr = A_6$ (11); $mp + qs = A_7$ (12); $n^2 + r^2 + t^2 = A_8$ (13); $np + rs + nt = A_9$ (14); $p^2 + s^2 + u^2 + v^2 = A_{10}$. (15)

From (6) $l = A_1^{\frac{1}{2}}$; hence from (7), (8) and (9) we find m, n and p; then from (10) we find q; and from (11) and (12) we can then find r and s; then from (13) and (14) we know t and u, and finally from (15) we find v. Thus equation (5) is always possible.

From $\Sigma(y^2) = (lb + mc + nd + pe)^2 + (qc + rd + se)^2 + (td + ue)^2 + (ve)^2 = 0$ we can assume $v^2e^2 = -(lb + mc + nd + pe)^2$ and $(td + ue)^2 = -(qc + rd + se)^2$, as the equation will then be satisfied. Therefore

$$ve = \sqrt{-1}(lb + mc + nd + pe), \tag{16}$$

$$td + ue = \sqrt{-1}\,(qc + rd + se). \tag{17}$$

From (16) and (17) we can express d and e by two equations of the *first* degree in b and c. Then, substituting in $\Sigma(y^3) = 0$, we have a homogeneous equation of the third degree in b and c. One of these may now be arbitrarily assumed and the other will then be found by the solution of an equation of the third degree.

If we make the above substitution in $\Sigma(y^4) = 0$ the final equation will be of the fourth degree. Therefore the general equation of the fifth degree can be reduced to either of the forms $x^5 + q_4x + q_5 = 0$ or $x^5 + q_3x^2 + q_5 = 0$.

In these two equations let $x = 1 \div z$. Substituting and reducing we get the two forms

$$z^5 + \frac{q_4}{q_5}z^4 + \frac{1}{q_5} = 0, \qquad \text{and } z^5 + \frac{q_3}{q_5}z^3 + \frac{1}{q_5} = 0.$$

This result was first obt'd by E. S. Bring and afterward by Mr. Jerrard.

Their method of making the substitutions may be seen in Todhunter's Theory of Equations or Serret's Cours D′ Algebre′ Superieure.

The above method is much easier than theirs, especially in finding the equation in y, which is there obtained by Tschirnhausen's substitution.

NOTE, BY MARCUS BAKER.—The Note on p. 53 of the last No. is an important one and I would very much like to see it made fuller.

Prob. 5, p. 66, Vol. I, has not been solved *in general*, the published solution applying only to a special case of the problem.

Prob. 49, p. 61, Vol. II, is erroneous and incomplete and needs further consideration.

Prob. 132, which you mention in your list, was completely solved by Prof. Hall at p. 55, Vol. IV. [This fact had escaped our recollection.—Ed.]